



On The Properties of Burr X-Topp Leone Distribution and Its Application

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ABSTRACT

The aim of this study is to establish some statistical properties and application of Burr X-Topp Leone distribution proposed by Usman et al., (2019), using an approach of Burr X family of generating continuous probability distributions. These properties obtained include survival and hazard functions, moments, order statistics, quantile function, probability weighted moment. Thus, the estimation parameters of the distribution were also derived using the method of maximum likelihood estimation technique. In order shows the importance of BXTLD over some existing methods, an application to a real dataset was analyzed and the result shows that BXTLD performed better in model fitting than some of the life time distributions considered. Therefore, this distribution can be considered as important alternatives to modeling real life datasets.

Keywords: Burr X G family, Topp-Leone, Burr X-Topp Leone, Moments

INTRODUCTION

The art of parameters induction to the baseline distribution has received a great deal of attention in recent years. The induction of one or more additional shape parameters to the baseline distribution makes the distribution more flexible especially for studying the tail properties (Tahir & Nadarajah, 2015).

Oguntunde *et al.* (2017) proposed Burr X-Exponential distribution using the Burr X family of distributions. The authors identified the basic statistical properties and the method of maximum likelihood in estimating the model parameters. The model was applied to three different real data sets to assess its flexibility over its baseline distribution.

Abouelmagd *et al.* (2018) defined and studied a new three-parameter lifetime model called the Burr X Frechet distribution. The new model has the advantage of being capable of modeling various shapes of aging and failure criteria. The authors derived some of its properties, which include ordinary and incomplete moments, quantile and generating functions, Renyi and η -entropies and order statistics. They utilized the method of maximum likelihood to estimate the model parameters. Also, simulation results were used to assess the performance of the maximum likelihood estimation. Therefore, they proved empirically the importance and flexibility of the new model comparing it with other extensions

of the Frechet distribution in the existing literature.

METHODOLOGY

In this study, we derived some properties of Burr X-Topp Leone Distribution proposed by Usman et al., (2019). In their work, they considered Burr X G family as generator and Topp-Leone distribution as a baseline.

Consider the both pdf and cdf of Topp-Leone distribution as:

$$g(x) = \beta(2 - 2x)(2x - x^2)^{\beta-1} \quad (1)$$

The associated cdf is

$$G(x) = (2x - x^2)^\beta \quad (2)$$

Where $\beta > 0$ is a shape parameter.

Also, the Burr X G family:

$$G(x) = [1 - \exp\{-\left(\frac{F(x)}{1-F(x)}\right)^2\}]^\theta \quad (3)$$

$$g(x) = \frac{2\theta f(x)F(x)}{[1-F(x)]^3} \exp\left[-\left(\frac{F(x)}{1-F(x)}\right)^2\right] \{1 - \exp\left[-\left(\frac{F(x)}{1-F(x)}\right)^2\right]\}^{\theta-1} \quad (4)$$

For $x > 0, \theta > 0$ (Oguntunde et al., 2017).

Received 1 September, 2021

Accepted 21 October, 2021

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Where, $G(x)$ and $g(x)$ are the cdf and pdf respectively.

However, the pdf proposed by Usman et al. (2019), gives:

$$g(x)_{BXTLD} = \frac{2\theta\beta(2-2x)(2x-x^2)^{2\beta-1}}{[1-(2x-x^2)^\beta]^3} \exp\left[-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\right] \left\{1 - \exp\left[-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\right]\right\}^{\theta-1} \quad (5)$$

Also, the cdf of BXTLD is given as:

$$G(x)_{BXTLD} = [1 - \exp\{-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\}]^\theta \quad (6)$$

Where, $x > 0, \theta > 0, \beta > 0$ are shape parameters.

Reliability Analysis for Burr X-Topp Leone Distribution

In this section, the expressions for the survival function and the hazard function of the BXTLD are derived. Survival function is represented in mathematical form as:

$$S(x) = 1 - G(x)_{BXTLD} \quad (7)$$

In this context, $G(x)_{BXTLD}$ is the cdf of BXTLD. Therefore, the expression for the survival function of the BXTLD is given as:

$$S(x) = 1 - [1 - \exp\{-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\}]^\theta \quad (8)$$

For $x > 0, \theta > 0, \beta > 0$

Also, the hazard function was derived using:

$$h(x) = \frac{g(x)_{BXTLD}}{S(x)} = \frac{2\theta\beta(2-2x)(2x-x^2)^{2\beta-1}}{[1-(2x-x^2)^\beta]^3} \exp\left[-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\right] \left\{1 - \exp\left[-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\right]\right\}^{\theta-1} \frac{\theta}{1 - [1 - \exp\{-\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2\}]^\theta} \quad (9)$$

Here, we adopted an idea of Yousof and Afify (2017) to provide a linear representation for the BXTLD

$$f(x) = \frac{2\theta f(x)F(x)}{[1-F(x)]^3} \exp\left\{-\left(\frac{F(x)}{1-F(x)}\right)^2\right\} \left[1 - \exp\left\{-\left(\frac{F(x)}{1-F(x)}\right)^2\right\}\right]^{\theta-1} \quad (10)$$

By simplifications, we have:

$$= 2\theta \sum_{k,l,m=0}^{\infty} \frac{(-1)^{k+l} \Gamma(\theta) \Gamma(2l+m+3) (k+1)^l}{k! l! m! \Gamma(\theta-k) \Gamma(2l+3) \Gamma(2l+3)} (2l+m+2) f(x) [F(x)]^{2l+m+1} \quad (11)$$

This can be written as

$$f(x) = \sum_{l,m=0}^{\infty} \pi_{l,m} \varphi_{2l+m+2}(x) \quad (12)$$

Where

$$\pi_{l,m} = \frac{2\theta(-1)^l \Gamma(\theta) \Gamma(2l+m+3)}{l! m! \Gamma(2l+3) \Gamma(2l+3)} \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)^l}{k! \Gamma(\theta-k)} \quad (13)$$

And

$$\varphi_{2l+m+2}(x) = (2l+m+2) f(x) [F(x)]^{2l+m+1} \quad (14)$$

$$M(x) = E(x^r) = \int_{-\infty}^{\infty} x^r f(x) dx \Rightarrow$$

$$\int_0^{\infty} x^r \sum_{l,m=0}^{\infty} \pi_{l,m} \varphi_{2l+m+2}(x) dx \quad (15)$$

Let

$$\sum_{l,m=0}^{\infty} \pi_{l,m} = A \quad (16)$$

Therefore

$$= A \int_0^{\infty} x^r f(x) [F(x)]^{2l+m+1} (2l+m+2) dx \quad (17)$$

Substitute the baseline (1 and 2) in 17

$$= A(2l+m+2) \int_0^1 x^r \beta(2-2x)(2x-x^2)^{\beta-1} [(2x-x^2)^\beta]^{2l+m+1} dx \quad (18)$$

Let $A(2l+m+2) = K$

$$= K \int_0^1 x^r \beta(2-2x)(2x-x^2)^{\beta(2+2l+m)-1} dx \quad (19)$$

$$= 2K\beta \int_0^1 x^{r+1} (1-x)(2-x)^{\beta(2+2l+m)-1} dx \quad (20)$$

Let $y = 1-x, x = 1-y, dx = -dy$

using the above terms in (20), we have:

$$= -2K\beta \int_1^0 (1-y)^{r+1} y(1+y)^{\beta(2+2l+m)-1} dy \quad (21)$$

$$= 2K\beta \int_0^1 (1-y)^{r+1} y(1+y)^{\beta(2+2l+m)-1} dy \quad (22)$$

If

$$(1+y)^{\beta(2+2l+m)-1} = \sum_{p=0}^{\infty} \binom{\beta(2+2l+m)-1}{p} y^p$$

Then (22) becomes

$$= \sum_{p=0}^{\infty} \frac{[\beta(2+2l+m)-1]!}{p! [\beta(2+2l+m)-1-p]!} y^p \quad (23)$$

$$= 2K\beta \sum_{p=0}^{\infty} \frac{\Gamma(\beta(2+2l+m))}{\beta! [\Gamma(\beta(2+2l+m)-p)]} \int_0^1 (1-y)^{r+1} y^{p+1} dy \quad (24)$$

$$\text{Let } 2\beta \sum_{p=0}^{\infty} \frac{\Gamma(\beta(2+2l+m))}{\beta! [\Gamma(\beta(2+2l+m)-p)]} = Q$$

$$= KQ \int_0^1 (1-y)^{r+1} y^{p+1} dy \quad (25)$$

$$E(x^r) = KQ \int_0^1 y^{p+1} (1-y)^{r+1} dy \tag{26}$$

$$= KQ \int_0^1 y^{(p+2)-1} (1-y)^{(r+2)-1} dy \tag{27}$$

$$= KQ\beta((p+2), (r+2)) \tag{28}$$

Mean of BXTLD

$$\mu_1 = E(x) = KQ\beta((p+2), (3)) \tag{29}$$

$$\mu_2 = E(x^2) = KQ\beta((p+2), (4)) \tag{30}$$

The 3rd and 4th moment of the BXTLD are given as:

$$\mu_3 = E(X^3) = KQ\beta[(p+2), (5)] \tag{31}$$

$$\mu_4 = E(X^4) = KQ\beta[(p+2), (6)] \tag{32}$$

Variance of BXTLD

$$Var(x) = E(x^2) - [E(x)]^2 \tag{33}$$

$$= KQ\beta((p+2), (4)) - [KQ\beta((p+2), (3))]^2 \tag{34}$$

Order Statistics of BXTLD

Order Statistics is an interesting topic in Statistics, suppose that X_1, X_2, \dots, X_n are independently and identically distributed random variables, each having a pdf and cdf of $f(x)$ and $F(x)$ respectively. If $Y_1 < Y_2 < \dots < Y_n$ represent the X_j 's is arranged in ascending order,

$$\frac{n!2\theta\beta(2-2x)(2x-x^2)^{2\beta[1+s]-1}}{(i-1)!(n-i)! [1-(2x-x^2)\beta]^3} \sum_{p,q,r,s=0}^{\infty} \binom{\theta-1}{p} \binom{n-1}{q} \binom{\theta(q+(i-1))}{r} (-1)^{p+q+r+s} \left[(\theta r(q+(i-1))) + (p+1) \right]^s \tag{37}$$

But

$$\left[1 - (2x - x^2)\beta \right]^{-[3+2s]} = \sum_{t=0}^{\infty} \frac{\Gamma(3+2s+t)}{t!\Gamma(3+2s)} ((2x - x^2))^{\beta t} \tag{38}$$

Thus,

$$= \frac{n!}{(i-1)!(n-i)!} 2\theta\beta(2-2x)(2x-x^2)^{\beta[2(1+s)+t]-1} \sum_{p,q,r,s=0}^{\infty} \binom{\theta-1}{p} \binom{n-1}{q} \binom{\theta(q+(i-1))}{r} (-1)^{p+q+r+s} \left[(\theta r(q+(i-1))) + (p+1) \right]^s \frac{\Gamma(3+2s+t)}{t!\Gamma(3+2s)} \tag{39}$$

Quantile Function of BXTLD

The quantile function is defined as the inverse of the cdf and it is given as: $Q(x) = F^{-1}(x)$.

From the CDF of BXTLD

then $Y_1 = \text{Min}(X_j)$ and it can be said that $Y_n = \text{Max}(X_j)$. In that sense, the Y_i 's are referred to as the order statistics of the Y_j 's Oguntunde (2017).

The distribution of i^{th} order statistics is given as:

$$f_{:n} = \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-1} \tag{35}$$

Consider

$$(2x - x^2)^n = \sum \binom{n}{q} (-1)^q (x^2)^q (2x)^{n-q}$$

Using binomial expansion series.

Applying pdf and cdf (5 and 6) of the BXTLD into 35

$$= \frac{n!2\theta\beta(2-2x)(2x-x^2)^{2\beta-1}}{(i-1)!(n-i)! [1-(2x-x^2)\beta]^3} \exp \left[- \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right] \left[1 - \exp \left\{ - \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right\} \right]^{\theta-1} \left[1 - \exp \left\{ - \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right\} \right]^{\theta(i-1)} \left[1 - \left[1 - \exp \left\{ - \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right\} \right]^{\theta} \right]^{n-1} \tag{36}$$

$$F(x) = \left[1 - \exp \left\{ - \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right\} \right]^\theta = U \tag{40}$$

$$U^{\frac{1}{\theta}} = 1 - \exp \left\{ - \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right\} \tag{41}$$

$$1 - U^{\frac{1}{\theta}} = \exp \left\{ - \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)\beta} \right)^2 \right\} \tag{42}$$

Taking the log of above equation

$$\log\left[1 - U^{\frac{1}{\theta}}\right] = -\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2 \quad (43)$$

$$-\log\left[1 - U^{\frac{1}{\theta}}\right] = -\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta}\right)^2 \quad (44)$$

Take the square root of both sides

$$-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right] = \frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \quad (45)$$

Crossly multiply

$$-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right] \left[1 - (2x - x^2)^\beta\right] = (2x - x^2)^\beta \quad (46)$$

From RHS divide all through by $(2x - x^2)^\beta$

$$-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right] = \frac{1}{\left[\frac{1}{(2x-x^2)^\beta} - 1\right]} \quad (47)$$

$$-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right] \left[\frac{1}{(2x-x^2)^\beta} - 1\right] = 1 \quad (48)$$

$$\frac{1}{(2x-x^2)^\beta} - 1 = \frac{1}{-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]} \quad (49)$$

$$\frac{1}{(2x-x^2)^\beta} = \frac{1}{-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]} + 1 \quad (50)$$

$$\frac{1}{(2x-x^2)^\beta} = \frac{\left[\frac{1}{-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]} + 1\right]}{1} \quad (51)$$

$$(2x - x^2)^\beta = \frac{1}{\left[\frac{1}{-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]} + 1\right]} \quad (52)$$

$$(2x - x^2) = \left[\frac{1}{-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]} + 1\right]^{\frac{1}{\beta}} \quad (53)$$

$$= \int_0^1 x^s \beta (2 - 2x) \sum_{n=0}^{\infty} \binom{\beta[(2j+k+1)+1]-1}{n} (-1)^n (2x)^{[\beta[(2j+k+1)+1]-1]-n} (x^2)^n \quad (64)$$

$$\text{Let } M = \left[\frac{1}{-\log\left[\left(1 - U^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]} + 1\right]^{\frac{1}{\beta}} \quad (54)$$

$$-x^2 + 2x - M = 0 \quad (54)$$

From quadratic equation formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (55)$$

Where $a = -1, b = 2$ and $c = -M$

Thus,

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(-1)(-M)}}{2(-1)} \quad (56)$$

$$x = \frac{-2 \pm \sqrt{4 - 4M}}{-2} \quad (57)$$

Therefore

$$x = 1 \pm \sqrt{1 - M} \quad (58)$$

Probability Weighted Moment

According to Yousof and Afify (2017) defined the probability weighted moment (PWM) as the expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly.

The $(s, r)th$ PWM of X following the Burr type X generator, say ρ_s, r , is formally defined by

$$P_{sr} = E[x^r F(x)^r] = \int_{-\infty}^{\infty} x^s F(x)^r f(x) dx \quad (59)$$

$$F(x)^r f(x) = \sum \Delta_{j,k} \pi_{2j+k+2}(x) = f(x)F(x) \quad (60)$$

$$E[Y^s_{2j+k+2}] = \int_{-\infty}^{\infty} Y^s_{2j+k+2}(x) \quad (61)$$

$$= \int_{-\infty}^{\infty} x^s f(x) [F(x)]^{2j+k+1} dx \quad (62)$$

Using pdf and cdf of the baseline

$$= \int_0^1 x^s \beta (2 - 2x) (2x - x^2)^{\beta-1} [(2x - x^2)^\beta]^{2j+k+1} dx \quad (63)$$

$$\begin{aligned} \text{Let } \sum_{n=0}^{\infty} \binom{\beta[(2j+k+1)+1]-1}{n} (-1)^n &= K \\ &= K \int_0^1 x^{s+2n} \beta(2-2x) (2x)^{[\beta[(2j+k+1)+1]-1]-n} \end{aligned} \tag{65}$$

$$= \beta K 2^{[\beta[(2j+k+1)+1]-1]-n} \int_0^1 x^{s+2n+[\beta[(2j+k+1)+1]-1]-n} \beta(2-2x) dx \tag{66}$$

Parameter Estimation

To estimate the parameters of the BXTL distribution, the method of Maximum Likelihood Estimation (MLE) was used as follows; Let $X = x_1, x_2, \dots, x_n$ be a random sample of n independently and identically distributed random variables each having the BXTL distribution, the likelihood function is given as:

$$L(x_1, x_2, \dots, x_n, \theta, \beta) = f(x_1, x_2, \dots, x_n, \theta, \beta) = \prod_{i=1}^n f(x_i, \theta, \beta) \tag{67}$$

$$\begin{aligned} L(x_1, x_2, \dots, x_n, \theta, \beta) &= \prod_{i=1}^n \left\langle \frac{2\theta\beta(2-2x)(2x-x^2)^{2\beta-1}}{[1-(2x-x^2)^\beta]^3} \exp \left[\left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right)^2 \right] \right. \\ &\quad \left. \exp \left[- \left(\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right)^2 \right] \right\rangle^{\theta-1} \end{aligned} \tag{68}$$

Take the natural logarithm of $L(x_1, x_2, \dots, x_n, \theta, \beta)$

$$\begin{aligned} \text{But let } L(x_1, x_2, \dots, x_n, \theta, \beta) &= L \\ L &= n \log(2) + n \log(\theta) + n \log(\beta) + \sum_{i=1}^n \log(2-2x) + (2\beta-1) \sum_{i=1}^n \log(2x-x^2) - 3 \sum_{i=1}^n \log[1-(2x-x^2)^\beta] - \sum_{i=1}^n \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 + (\theta-1) \sum_{i=1}^n \log \left[1 - \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\} \right] \end{aligned} \tag{69}$$

Differentiating L with respect to parameters θ gives:

$$\frac{dL}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n \log \left[1 - \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\} \right] \tag{70}$$

Differentiating L w.r.t. β .

$$\frac{dL}{d\beta} = \frac{n}{\beta} + 2 \sum_{i=0}^n \log(2x-x^2) - 3 \sum_{i=1}^n \log[1-(2x-x^2)^\beta] -$$

$$\sum_{i=1}^n \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 (\theta-1) \sum_{i=1}^n \log \left[1 - \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\} \right] \tag{71}$$

Where from (71)

$$\log[1-(2x-x^2)^\beta] = \frac{-\log(2x-x^2) \exp\{\beta \log(2x-x^2)\}}{1-(2x-x^2)^\beta} \tag{72}$$

Also, from 71

$$\begin{aligned} \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 &= (2x-x^2)^{2\beta} \log(2x-x^2)^2 [1-(2x-x^2)^\beta]^{-1} + (2x-x^2)^{2\beta} \log(2x-x^2) (2x-x^2)^\beta [1-(2x-x^2)^\beta]^{-2} \end{aligned} \tag{73}$$

From 71

$$\begin{aligned} \log \left[1 - \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\} \right] &= \frac{\exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\}}{1 - \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\}} (2x-x^2)^{2\beta} \log(2x-x^2)^2 [1-(2x-x^2)^\beta]^{-1} + (2x-x^2)^{2\beta} \log(2x-x^2) (2x-x^2)^\beta [1-(2x-x^2)^\beta]^{-2} \end{aligned} \tag{74}$$

Using 72, 73 and 74 in 71

$$\begin{aligned} \frac{dL}{d\beta} &= \frac{n}{\beta} + 2 \sum_{i=1}^n \log(2x-x^2) + 3 \sum_{i=1}^n [\log(2x-x^2)] (2x-x^2)^\beta [1-(2x-x^2)^\beta]^{-1} - \sum_{i=1}^n (2x-x^2)^{2\beta} \log(2x-x^2) [1-(2x-x^2)^\beta]^{-2} + (\theta-1) \sum_{i=1}^n \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\} \left[1 - \exp \left\{ - \left[\frac{(2x-x^2)^\beta}{1-(2x-x^2)^\beta} \right]^2 \right\} \right]^{-1} (2x-x^2)^{2\beta} \log(2x-x^2) [1-(2x-x^2)^\beta]^{-2} \end{aligned} \tag{75}$$

Meanwhile, the result in (75) cannot be obtained analytically except numerically from suitable statistical software when data sets are available.

RESULTS AND DISCUSSION

In this section, the goodness-of-fits of the distributions derived in this research have been discussed using the maximum likelihood estimate with an application to real life dataset (obtained from National Bureau of Statistics which is the Proportion of Internal Generating Revenue of year 2020 by 36 States and FCT). The likelihood functions do not seem to be

solved analytically because they are not in closed form. The equations can be solved iteratively using optimum in R-software till sufficiently close values are obtained (R Core Team, 2018). Thus, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), and Hannan Quin Information Criterion (HQIC) have been computed as follows to compare the distributions performance with some existing distributions in model fitting. Thus, the distribution with the minimum value of these criteria is considered the best.

Table 1: Parameters, AIC, BIC, CAIC and HQIC of BXTLD, TPL, Burr X, WBX and WE

| Models | Parameters | Estimate | AIC | BIC | CAIC | HQIC |
|--------|------------|----------|-----------|-----------|-----------|-----------|
| TPL | β | 0.28688 | -138.6789 | -137.068 | -138.5646 | -138.111 |
| Burr X | θ | 8.08829 | 163.509 | 160.2872 | 163.1561 | 162.3732 |
| | λ | 0.22095 | | | | |
| | | | | | | |
| WBX | α | 1.02144 | -117.9651 | -111.5215 | -116.7151 | -115.6934 |
| | β | 0.62113 | | | | |
| | θ | 1.36595 | | | | |
| | λ | 1.49687 | | | | |
| WE | α | 1.78810 | 2065.184 | 2058.741 | 2063.934 | 2062.913 |
| | c | 0.95206 | | | | |
| | λ | 0.70096 | | | | |
| | γ | 0.00131 | | | | |
| BXTLD | θ | 0.43837 | -150.1105 | -146.8887 | -149.7576 | -148.9747 |
| | β | 0.40172 | | | | |

The above table shows the parameter estimates of the derived BXTLD distribution compared to the existing distributions considered (TPL, Burr X, WBX and WE) using dataset of proportion of Nigeria Internal Generated Revenue of the year 2020. The result revealed that the new derived BXTLD distribution has the minimum values of AIC, BIC, CAIC and HQIC compared with the existing distributions considered, this implies that based on the information selection criteria the new distribution performed better in terms of model fitting.

Conclusion

The current study intends to establish some properties and application of Burr X-Topp Leone distribution proposed by Usman et al. (2019). In conclusion, some statistical properties of the work of Burr X-Topp Leone were established, which are moment, order statistics, quantile function, mean and variance,

probability weighted moment and reliability analysis were obtained. In addition, estimators for the parameters of the distribution were derived by method of maximum likelihood. Thus, based on the AIC, BIC, CAIC and HQIC the results from this study revealed that BXTLD performed better in model fitting than some of the life time distributions considered. Therefore, this distribution can be considered as important alternatives to modeling real life datasets.

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